DISRUPTION OF POTENTIAL GAS FLOWS ADJACENT TO THE REGION OF REST

PMM Vol. 35, №3, 1971, pp. 482-491
A. F. SIDOROV
(Sverdlovsk)
(Received July 31, 1970)

The paper deals with the problem of motion of a plane and a three-dimensional piston of arbitrary, sufficiently smooth shape, through a gas at rest, when the piston has zero normal initial velocity and nonzero normal acceleration. Solutions in the neighborhood of curvilinear weak discontinuities which detach from the piston at the initial instant and move through the gas at rest, are given in an approximate form. Exact formulas are obtained for the limiting times of existence of smooth potential flows near the weak discontinuities and their dependence on the geometry of the piston, and on the magnitude of the prescribed acceleration under the assumption that the weak discontinuity is not overtaken by the resulting perturbations. Certain properties of the flows near the weak discontinuities are studied.

1. It is a well known fact that, when a piston is advanced into a homogeneous polytropic gas at rest contained in a semi-infinite rectilinear channel $(x \geqslant 0)$ according to the rule $x = f(t)$, its motion beginning at the instant $t = 0$ with zero initial velocity and positive initial acceleration $(f(0) = f'(0) = 0,\ f''(0) > 0)$, then the smooth solution between the piston and the weak discontinuity moving through the stationary gas with the speed of sound, will exist only for a limited period of time [1]. The compression wave formed is the Riemann wave and a shock wave will appear in the flow at some $t = t^* > 0$. If the infinite gradients of the gasdynamic quantities appear directly at the line of weak discontinuity (this will occur e. g. when the piston moves according to the law $x = at^2$ where the acceleration $a > 0$ is constant), then the instant t^* of disruption of the corresponding Riemann wave is easily found to be

$$t^* = \frac{1}{(\gamma+1)a} = \frac{2}{(\gamma+1)|W_0|} \tag{1.1}$$

where γ is the adiabatic index in the gas equation of state and W_0 is the acceleration of the piston at $t = 0$.

We shall consider a more general problem of determining the time t^* of the onset of disruption of the potential flow occurring directly on the surface of weak discontinuity, and of constructing an approximate solution near the weak discontinuity for the case when pistons of arbitrary shape advance into the gas and the resulting flows are two- or three-dimensional.

Let the (sufficiently smooth) surface S_0 divide a three-dimensional x_1, x_2, x_3-space into two parts, one of which is filled with a homogeneous polytropic gas at rest, and let the speed of sound in this part be equal to $c \equiv 1$. At the instant $t = 0$ the piston S_t begins to advance into the gas (the surface S_0 corresponds to the initial position of the piston) according to some law in such a manner, that at $t = 0$ the normal velocity of motion V_n is zero and the normal acceleration W_n is nonzero everywhere. It is clear that a compression wave will begin to move through the unperturbed gas. This wave will be bounded on one side by the piston surface S_t, and on the other side by the surface R_t of the weak discontinuity moving with a unit normal velocity across the gas at rest. The form of the surface R_t will depend only on the geometry of S_0 The flow will be isentropic and potential until the moment when strong discontinuities appear within it.

The flow potential $\Phi\ (x_1,\ x_2,\ x_3,\ t)$ will satisfy, in the general three-dimensional case, the equation (1.2)

$$\Phi_{tt} + 2\sum_i \Phi_{x_i}\Phi_{x_i t} + 2\sum_{ik}(1-\delta_{ik})\,\Phi_{x_i}\Phi_{x_k}\Phi_{x_i x_k} - \sum_i(\Theta - \Phi_{x_i}{}^2)\,\Phi_{x_i x_i} = 0$$

where the subscripts accompanying Φ denote differentiation with respect to the indicated variables, δ_{ik} is the Kronecker delta and

$$\Theta = c^2 = \frac{1}{\varkappa}\Big(M - \Phi_t - \frac{1}{2}\sum \Phi_{x_i}{}^2\Big), \quad \varkappa = \frac{1}{\gamma - 1}, \quad M = \mathrm{const} \tag{1.3}$$

We shall study the behavior of the solutions of the stated problem near the surface R_t along which we have $\Phi_{x_i} = u_i = 0$ where u_i denote the components of the velocity vector $\mathbf{u}$.

Note 1.1. The problem of adjacency of unstable plane and three-dimensional gas flows to a region of rest across a weak discontinuity was studied in [2, 3]. However, only the case of rarefaction waves (withdrawal of pistons) was studied in these papers and only the class of double waves was considered to construct a solution near the weak discontinuity. This led to restricting the form of the surface R_t which had to be developable at any instant of time.

To begin with we shall study the case of a plane, parallel unsteady motion when the subscripts in (1.2) are $i,\ k = 1,\ 2$. In the following we shall use for convenience the independent variables t, u_1, u_2 in (1.2) (assuming that u_1 and u_2 are functionally independent near R_t). The change of variables is easily effected with the help of the Legendre transformations, using the function $\Psi\ (u_1,\ u_2,\ t)$ given by

$$\Psi = x_1 u_1 + x_2 u_2 - \Phi + Mt \tag{1.4}$$

The final expression for $\Psi\ (u_1,\ u_2,\ t)$ is

$$\Psi_{tt}\,(\Psi_{11}\Psi_{22} - \Psi_{12}{}^2) + [\Theta - (\Psi_{1t} - u_1)^2]\Psi_{22} + \\ + 2\,(\Psi_{1t} - u_1)\,(\Psi_{2t} - u_2)\,\Psi_{12} + [\Theta - (\Psi_{2t} - u_2)^2]\Psi_{11} = 0 \tag{1.5}$$

and we have (1.6)

$$\Psi_{ik} = \frac{\partial^2\Psi}{\partial u_i \partial u_k}, \quad \Psi_{it} = \frac{\partial^2\Psi}{\partial u_i \partial t}, \quad \Theta = \frac{1}{\varkappa}\Big[\Psi_t - \frac{1}{2}\,(u_1{}^2 + u_2{}^2)\Big], \quad x_i = \Psi_i$$

(Eq. (1.5) was already obtained in [4]).

Since a point corresponding to the region of rest in the hodograph $u_1,\ u_2$-plane is the point (0, 0), it is expedient to pass in (1.5) and (1.6) to the polar coordinates $u_1 = r\cos\varphi$ and $u_2 = r\sin\varphi$. Then the boundary separating the perturbed motion from the region of rest will have a corresponding line in the $r,\ \varphi$-variables, given by $r = 0$. The equation for $\Psi\ (r,\ \varphi,\ t)$ has the form

$$\Psi_{tt}\,(-r^{-2}\Psi_{r\varphi}{}^2 - r^{-4}\Psi_\varphi{}^2 + r^{-2}\Psi_{\varphi\varphi}\Psi_{rr} + r^{-1}\Psi_r\Psi_{rr} + 2r^{-3}\Psi_{r\varphi}\Psi_\varphi) + \\ + \varkappa^{-1}(\Psi_t - {}^1/_2 r^2)\,[\Psi_{rr} + r^{-1}\Psi_r + r^{-2}\Psi_{\varphi\varphi}] - [r^{-2}\Psi_{rt}{}^2\Psi_{\varphi\varphi} + r^{-1}\Psi_{rt}{}^2\Psi_r] +$$

$$+ 2r^{-2}\Psi_{rt}\Psi_{\varphi t}\Psi_{r\varphi} - 2r^{-3}\Psi_{rt}\Psi_{\varphi t}\Psi_{\varphi} - r^{-2}\Psi_{\varphi t}{}^2\Psi_{rr} + 2\left([r^{-1}\Psi_{rt}\Psi_{\varphi\varphi} + \right. \qquad (1.7)$$
$$\left. + \Psi_{rt}\Psi_r] - r^{-1}\Psi_{\varphi t}\Psi_{r\varphi} + r^{-2}\Psi_{\varphi t}\Psi_{\varphi}\right) - \Psi_{\varphi\varphi} - r\Psi_r = 0$$

while (1.6) is replaced ($i = 1, 2$), respectively, by

$$x_1 = \Psi_r \cos\varphi - r^{-1}\Psi_\varphi \sin\varphi, \qquad x_2 = \Psi_r \sin\varphi + r^{-1}\Psi_\varphi \cos\varphi \qquad (1.8)$$

When $r = 0$, Eqs. (1.8) determine the motion of the weak discontinuity, consequently the limit $\lim r^{-1}\Psi_\varphi = \Pi(\varphi, t)$ as $r \to 0$ must exist.

We further assume that near the surface R_t the fourth order derivatives of $\Psi(r, \varphi, t)$ in which the differentiation has been performed twice with respect to each independent variable, are all continuous. This assumption of the smoothness of the flow near R_t is in fact realized in a number of real flows, e. g. in one-dimensional flows, provided that the piston moves sufficiently smoothly [1], or in the class of plane and three-dimensional double waves [2, 3]. In particular, the assumption just made implies that the perturbations in the flow characterized by first order discontinuities in u_1, u_2 and c do not overtake the weak discontinuity $r = 0$.

Let us write the function $\Psi(r, \varphi, t)$ in the form

$$\Psi(r, \varphi, t) = \Psi(0, \varphi, t) + \Psi_r(0, \varphi, t)\, r + {}^1\!/_2 r^2\, \Psi_{rr}(r_1, \varphi, t) \qquad (1.9)$$
$$0 < r_1 < r$$

Assuming $\Psi_r(0, \varphi, t) = \Gamma(\varphi, t)$ and utilizing the previous assumptions we find, that $\Pi(\varphi, t) = \Gamma_\varphi\ (\Psi_\varphi(0, \varphi, t) = 0)$. The motion of the weak discontinuity will therefore be described by the following equations obtained from (1.8)

$$x_1 = \Gamma\cos\varphi - \Gamma_\varphi \sin\varphi, \qquad x_2 = \Gamma\sin\varphi + \Gamma_\varphi\cos\varphi \qquad (1.10)$$

Computing the normal velocity of motion of the weak discontinuity (1.10) (which by definition is equal to unity) we arrive at the condition $\Gamma_t = 1$ (generally speaking the condition $|\Gamma_t| = 1$ should be obtained, but taking into account that t can also assume negative values makes it possible to limit oneself to the case $\Gamma_t = 1$ without loss of generality). Thus we have

$$\Gamma(\varphi, t) = t + f(\varphi) \qquad (1.11)$$

Here $f(\varphi)$ is an arbitrary function which can be used in assigning an arbitrary form to the weak discontinuity at $t = 0$ in the following manner

$$x_1 = f(\varphi)\cos\varphi - f'(\varphi)\sin\varphi, \qquad x_2 = f(\varphi)\sin\varphi + f'(\varphi)\cos\varphi \qquad (1.12)$$

Let us now estimate the order of all the terms of (1.7) near $r = 0$, using the representation (1.9) together with the analogous representations for the first and second derivatives of $\Psi(r, \varphi, t)$. For the first order derivatives we obtain

$$\Psi_\varphi = rf'(\varphi) + {}^1\!/_2 r^2\Psi_{\varphi rr}(r_2, \varphi, t), \qquad \Psi_r = t + f(\varphi) + r\Psi_{rr}(r_3, \varphi, t) \quad (1.13)$$
$$\Psi_t = K_1 + r + {}^1\!/_2 r^2\Psi_{rrt}(r_4, \varphi, t), \qquad K_1 = \text{const}$$

(all last terms in the Taylor expansions in powers of r contain derivatives of $\Psi(r, \varphi, t)$ in which the differentiation with respect to r. $0 < r_k < r$) has been performed twice). Similar formulas are easily obtained for the second order partial derivatives of $\Psi(r, \varphi, t)$ with respect to all its variables (for Ψ_{rr} no expansion is necessary). Retaining in (1.7) the terms of the $O(1)$-order and neglecting the terms of the $o(1)$-order we obtain the following approximate relation:

$$\Psi_{rr}(r,\varphi,t)+(\gamma+1)(t+f+f'')-2(t+f+f'')\Psi_{rrt}(r_5,\varphi,t)\approx 0 \quad (1.14)$$

We find here that the only terms of (1.7) contributing to (1.14) are those contained within the square brackets. Setting in (1.14)

$$r=0, \qquad \Psi_{rr}(0,\varphi,t)=T(\varphi,t)$$

we obtain an exact differential equation for determining the structure of $T(\varphi, t)$

$$T+(\gamma+1)(t+f+f'')-2(t+f+f'')T_t=0 \qquad (1.15)$$

The latter equation is easily integrated, yielding the following expression for the function $\Psi_{rr}(0,\varphi,t)$:

$$\Psi_{rr}(0,\varphi,t)=C(\varphi)(t+f+f')^{1/2}+(\gamma+1)(t+f+f'') \qquad (1.16)$$

where $C(\varphi)$ is an arbitrary function.

It should be noted that in the t, r, φ-space the plane $r=0$ will be a characteristic surface for (1.7) and that the Cauchy data will determine the function $\Psi_{rr}(0,\varphi,t)$ nonuniquely when $r=0$. The function $C(\varphi)$ can be defined from the condition that the normal acceleration of the piston at the time $t=0$ is given. Indeed, obtaining the derivative $\partial r/\partial t$ at $r=0$ from (1.8) we find, that either

$$\Psi_{rt}+\Psi_{rr}\frac{\partial r}{\partial t}=0 \quad \text{for} \quad r=0$$

or

$$\frac{\partial r}{\partial t}=\cos\varphi\frac{\partial u_1}{\partial t}+\sin\varphi\frac{\partial u_2}{\partial t}=-\frac{1}{\Psi_{rr}(0,\varphi,t)} \qquad (1.17)$$

Let us set $\partial r/\partial t=-W(\varphi)$ when $r=0$ and $t=0$. From (1.17) it follows that $-W(\varphi)$ is the value of the normal acceleration of the piston at $t=0$, since we know [3] that the instantaneous streamlines of the perturbed flow are always orthogonal to the surface of the weak discontinuity moving across the region of rest, and the vector $(\cos\varphi, \sin\varphi)$ is orthogonal to R_t. Thus using (1.16) and (1.17) we obtain

$$C(\varphi)=(f+f'')^{-1/2}(W^{-1}(\varphi)-(\gamma+1)(f+f'')) \qquad (1.18)$$

We have the following approximate representation for the function $\Psi(r,\varphi,t)$ near R_t (we replace ($\Psi_{rr}(r_1,\varphi,t)$ in (1.9) by $\Psi_{rr}(0,\varphi,t)$):

$$\Psi(r,\varphi,t)=K_0+K_1t+(t+f(\varphi))r+{}^1/_2[C(\varphi)(t+f+f'')^{1/2}+$$
$$+(\gamma+1)(t+f+f'')]r^2 \qquad (K_0, K_1=\text{const}) \qquad (1.19)$$

where $C(\varphi)$ is given by (1.18). In this connection, the flow in the physical x_1, x_2, t-space is constructed according to the formulas

$$x_1=\{t+f(\varphi)+r[C(\varphi)(t+f+f'')^{1/2}+(\gamma+1)(t+f+f'')]\}\cos\varphi-$$
$$-\Big\{f'(\varphi)+\frac{1}{2}r\Big[C'(\varphi)(t+f+f'')^{1/2}+C(\varphi)\frac{f'+f'''}{2(t+f+f'')^{1/2}}+$$
$$+(\gamma+1)(f'+f''')\Big]\Big\}\sin\varphi$$
$$x_2=\{t+f(\varphi)+r[C(\varphi)(t+f+f'')^{1/2}+(\gamma+1)(t+f+f'')]\}\sin\varphi+$$
$$+\Big\{f'(\varphi)+\frac{1}{2}r\Big[C'(\varphi)(t+f+f'')^{1/2}+C(\varphi)\frac{f'+f'''}{2(t+f+f'')^{1/2}}+$$
$$+(\gamma+1)(f'+f''')\Big]\Big\}\cos\varphi \qquad (1.20)$$

We note that the expression $t+f+f''=R(\varphi,t)$ represents the radius of curvature of the surface R_t (or of some curve on the plane $t = \text{const}$). This follows at once from (1.20) when $r = 0$ (we assume that $f + f'' > 0$).

The instant t^* of the beginning of disruption of the potential flow can be determined directly on R_t by finding the numerically smallest root of the equation $\Psi_{rr}(0, \varphi, t) = 0$ (the derivative $\partial r / \partial t$ then becoming infinite). The final expressions for t^* are as follows: for

$$t^* < 0, W(\varphi) > 0, W(\varphi)(f + f'') > 1/(\gamma + 1)$$

we have

$$t^* = -\min_\varphi \left\{ f + f''; \frac{1}{(\gamma+1)^2 W(\varphi)} \left[2(\gamma + 1) - \frac{1}{W(\varphi)(f + f'')} \right] \right\} \tag{1.21}$$

while for

$$t^* > 0, \quad W(\varphi) < 0$$

we have

$$t^* = \min_\varphi \frac{-1}{(\gamma+1)^2 W(\varphi)} \left[2(\gamma + 1) - \frac{1}{W(\varphi)(f + f'')} \right] \tag{1.22}$$

Using the formulas (1.21) and (1.22) we can also obtain easily the position at which the potential flow begins to break down. If the minimum value of t^* is reached according to (1.21) for $t^* = -\min_\varphi (f + f'')$, then the radius of curvature becomes zero at some point of R_t (in the one-dimensional case this corresponds to the focusing of the weak discontinuity on the axis of symmetry).

Let us consider in more detail the case of a cylindrical piston of radius R_0 which begins to move into the gas with a constant acceleration $W_0 > 0$. In accordance with formulas (1.21) and (1.22) we obtain

$$|t^*| = \left[2(\gamma + 1) - \frac{1}{W_0 R_0} \right] \frac{1}{W_0 (\gamma + 1)^2} \quad \text{when } W_0 R_0 > \frac{1}{\gamma + 1}$$

$$|t^*| = R_0 \quad \text{when } W_0 R_0 \leqslant \frac{1}{\gamma + 1} \tag{1.23}$$

for the case when the gas is contained within an infinite cylindrical tube whose walls begin to move inwards, and

$$t^* = \left[2(\gamma + 1) + \frac{1}{W_0 R_0} \right] \frac{1}{W_0 (\gamma + 1)^2} \tag{1.24}$$

for the case of a cylindrical piston moving into the gas contained outside a cylindrical tube of radius R_0.

Passing in (1.23) and (1.24) to the limit, W_0 being kept fixed and $R_0 \to \infty$ (this corresponds to the case when the piston surface is a plane at $t = 0$), we obtain

$$\lim_{R_0 \to \infty} t^* = \frac{2}{(\gamma + 1) W_0}$$

which agrees with (1.1). Addition and subtraction in (1.23) and (1.24) of the term $W_0^{-2} R_0^{-1} (\gamma + 1)^{-2}$ gives the corresponding quantitative correction in t^* for the cylindrical effect, for the cases indicated above.

Note 1.2. Estimating the order of the terms in (1.7) makes it possible to obtain simplified model equations valid in the neighborhood of R_t. In particular

$$r\Psi_{rr} - 2(\Psi_{\varphi\varphi} + r\Psi_r)\Psi_{rrt} \pm (\gamma + 1)(\Psi_{\varphi\varphi} + r\Psi_r) = 0$$

Such equations can be employed both to construct approximate solutions, and to analyze the correctness of the formulation of various boundary value problems for (1.7) with the parameters given on R_t.

Note 1.3. In the one-dimensional case ($f(\varphi) = \text{const}$, $W(\varphi) = \text{const}$), for values of t close to t^*, the region of application of the approximate solutions (1.19) and (1.20) (the neighborhood of the surface R_t on the x_1x_2-plane) contracts and degenerates into a line (this follows from (1.20)), when $t = t^*$. This is to be expected, as the approximate solution (1.20) was constructed using only such dynamic characteristics of the motion of the piston, as the velocity and acceleration. The correct transfer of the velocity distribution profiles near R_t can be achieved for the values of t close to t^* only by taking into account the dynamic characteristics of the order higher than that of acceleration, i.e. in the expression for $\Psi(r, \varphi, t)$ the term of the order $O(r^3)$ must be taken into account. This term can be computed by employing an analogous approach based on the assumption of sufficient smoothness of the flow near R_t. In the general two-dimensional case, the same contraction of the region of applicability of the formulas (1.20) takes place at t close to t^* in the neighborhood of a point on the x_1x_2-plane corresponding to the point (φ^*, t^*) at which $\partial r/\partial t$ becomes infinite.

2. Let us consider the motion near the surface R_t for the three-dimensional space. As in the plane case, we introduce the function $\Psi(u_1, u_2, u_3, t)$ given by

$$\Psi = x_1u_1 + x_2u_2 + x_3u_3 - \Phi + Mt \tag{2.1}$$

and obtain the following equations for Ψ:

$$\Psi_{tt}\Delta + \sum_{ik}[2u_k\Psi_{ti} - 2u_iu_k(1-\delta_{ik}) + (c^2 - u_k^2)\delta_{ik} - \Psi_{ti}\Psi_{tk}]\Delta_{ik} = 0 \tag{2.2}$$

$$x_i = \Psi_i \tag{2.3}$$

$$\left(c^2 = \varkappa^{-1}\left[\Psi_t - \frac{1}{2}(u_1^2 + u_2^2 + u_3^2)\right]\right) \tag{2.4}$$

$$\Delta = \det\|\Psi_{\mu\nu}\| \quad (\mu, \nu = 1, 2, 3), \qquad \Delta_{ik} = (-1)^{i+k}\begin{vmatrix}\Psi_{mp} & \Psi_{np}\\ \Psi_{mq} & \Psi_{nq}\end{vmatrix}$$

$$\begin{pmatrix} m, n \neq k; & m < n\\ p, q \neq i; & p < q\end{pmatrix}$$

Introducing the polar coordinates

$$u_1 = r\cos\varphi\sin\theta, \qquad u_2 = r\sin\varphi\sin\vartheta, \qquad u_3 = r\cos\theta$$

we can write (2.3) in the form

$$\begin{aligned} x_1 &= \Psi_r\cos\varphi\sin\theta - \Psi_\varphi\frac{\sin\varphi}{r\sin\theta} + \Psi_\theta\frac{\cos\varphi\cos\theta}{r}\\ x_2 &= \Psi_r\sin\varphi\sin\theta + \Psi_\varphi\frac{\cos\varphi}{r\sin\theta} + \Psi_\theta\frac{\sin\varphi\cos\theta}{r}\\ x_3 &= \Psi_r\cos\theta - \Psi_\theta\frac{\sin\theta}{r}\end{aligned} \tag{2.5}$$

When $r = 0$, Eqs. (2.5) determine the motion of the weak discontinuity surface R_t provided that, as in the plane case, the following limits exist

$$\lim_{r\to 0} r^{-1}\Psi_\varphi = \Gamma_\varphi(\varphi, \theta, t), \qquad \lim_{r\to 0} r^{-1}\Psi_\theta = \Gamma_\theta(\varphi, \theta, t)$$

where

$$\Gamma(\varphi, \theta, t) = \Psi_r(0, \varphi, \theta, t)$$

Further assumption of continuity of all fourth order derivatives of $\Psi(r, \varphi, \theta, t)$ containing second order differentiation with respect to any of its arguments yields, in analogy with Sect. 1, the following result. $\Gamma_t = 1$ and $\Gamma = t + P(\varphi, \theta)$ where $P(\varphi, \theta)$ is an arbitrary function determining, at $t = 0$, the form of the surface S_0

$$\Psi(r, \varphi, \theta, t) = \Psi(0, \varphi, \theta, t) + r\Gamma + \frac{r^2}{2}\Psi_{rr}(r^\circ, \varphi, \theta, t) \qquad (2.6)$$

$$(0 < r^\circ < r)$$

The motion of R_t is determined by the following equations:

$$x_1 = (t + P(\varphi, \theta))\cos\varphi\sin\theta - P\frac{\sin\varphi}{\sin\theta} + P_\theta\cos\varphi\cos\theta$$

$$x_2 = (t + P(\varphi, \theta))\sin\varphi\sin\theta + P_\varphi\frac{\cos\varphi}{\sin\theta} + P_\theta\sin\varphi\cos\theta$$

$$x_3 = (t + P(\varphi, \theta))\cos\theta - P_\theta\sin\theta \qquad (2.7)$$

The approximate solution near R_t as well as $\Psi_{rr}(0, \varphi, \theta, t)$ could both be obtained by exactly the same method as that in Sect. 1, but this approach is awkward. It appears that the function $\Psi_{rr}(0, \varphi, \theta, t)$ can be easily found using the results of [3] concerning the problem of propagation of a weak discontinuity along the bicharacteristics covering the characteristic surface R_t.

For the jumps in the values of the directional derivatives of u_i and c on R_t, the following relation holds [3]:

$$-\left(\left[\frac{\partial u_1}{\partial t}\right], \left[\frac{\partial u_2}{\partial t}\right], \left[\frac{\partial u_3}{\partial t}\right], \left[\frac{\partial c}{\partial t}\right]\right) = \sigma \mathbf{l} \qquad (2.8)$$

where $\mathbf{l}$ is the right null vector of the characteristic matrix of the system of gasdynamic equations, the symbol $[\partial f / \partial t]$ denotes the difference between the limit values of the derivatives $\partial f / \partial t$ taken at each side of R_t and σ in a scalar function which can be obtained along a fixed bicharacteristic from the ordinary differential "transport" equation of the jumps in the directional derivatives. It was shown in [3] that, when R_t moves across the region of rest, the bicharacteristics are straight lines and along a fixed bicharacteristic ($\varphi = \text{const}$, $\theta = \text{const}$) σ has the form

$$\sigma = \{\sqrt{t + C_1}\sqrt{t + C_2}[(\gamma + 1)\ln(\sqrt{t + C_1} + \sqrt{t + C_2}) + C_3]\}^{-1} \qquad (2.9)$$

where time is taken as the parameter on the bicharacteristic. Here $t + C_1$ and $t + C_2$ are the radii of curvature of the principal normal cross sections of the surface R_t, and C_3 is an arbitrary constant. As $\mathbf{l}$ we can take

$$\mathbf{l} = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta, {}^1/_2(\gamma - 1)) \qquad (2.10)$$

Relations (2.8) and (2.10) yield the following value for the derivative $\partial r / \partial t$ in the perturbed motion when $r = 0$:

$$\partial r / \partial t = -\sigma \quad \text{for} \quad r = 0 \qquad (2.11)$$

On the other hand, using the assumptions made earlier about the function Ψ, we find the following expression for $\partial r / \partial t \, (r = 0)$ from (2.5)

$$\frac{\partial r}{\partial t} = -\frac{1}{\Psi_{rr}(0, \varphi, \theta, t)} \qquad (2.12)$$

Thus we have

$$\Psi_{rr}(0, \varphi, \theta, t) = \sqrt{t + R_1(\varphi, \theta)}\sqrt{t + R_2(\varphi, \theta)}\,[(\gamma + 1) \times$$

$$\times \ln(\sqrt{t + R_1(\varphi, \theta)} + \sqrt{t + R_2(\varphi, \theta)}) + C_3(\varphi, \theta)] \qquad (2.13)$$

where $R_1(\varphi, \theta)$ and $R_2(\varphi, \theta)$ are the radii of curvature of the principal normal cross sections of the surface S_0, which are easily obtained provided that the function $P(\varphi, \theta)$ is given and $C_3(\varphi, \theta)$ is arbitrary. The latter can be expressed, as it was done in Sect. 1, in terms of the normal acceleration $W(\varphi, \theta)$ of the surface S_0 at $t = 0$.

The final approximate expression for the function Ψ near R_t is

$$\Psi(r, \varphi, \theta, t) = K_0 + K_1 t + (t + P(\varphi, \theta)) r + {}^1/_2 r^2 \sqrt{t + R_1(\varphi, \theta)} \times \\ \times \sqrt{t + R_2(\varphi, \theta)} \left[(\gamma + 1) \ln\left(\sqrt{t + R_1(\varphi, \theta)} + \sqrt{t + R_2(\varphi, \theta)}\right) + C(\varphi, \theta)\right] \tag{2.14}$$

Passing to the variables x_i and t is effected by means of the formulas

$$\begin{aligned} x_1 &= (t + P + rL)\cos\varphi\sin\theta - (\Phi_\varphi + {}^1/_2 rL_\varphi)\sin\varphi/\sin\theta + \\ &\quad + (\Phi_\theta + {}^1/_2 rL_\theta)\cos\varphi\cos\theta \\ x_2 &= (t + P + rL)\sin\varphi\sin\theta + (\Phi_\varphi + {}^1/_2 rL_\varphi)\cos\varphi/\sin\theta + \\ &\quad + (\Phi_\theta + {}^1/_2 rL_\theta)\sin\varphi\cos\theta \\ x_3 &= (t + P + rL)\cos\theta - (\Phi_\theta + {}^1/_2 rL_\theta)\sin\theta \end{aligned} \tag{2.15}$$

where

$$L(t, \varphi, \theta) = \Psi_{rr}(0, \varphi, \theta, t), \qquad C(\varphi, \theta) = \frac{1}{W\sqrt{R_1 R_2}} - (\gamma+1)\ln\left(\sqrt{R_1} + \sqrt{R_2}\right) \tag{2.16}$$

Finding the smallest root of the equation $L(t^*, \varphi, \theta) = 0$, we obtain the following expression for t^*:

$$t^* = -\min_{\varphi,\theta}\{R_1(\varphi, \theta), \ R_2(\varphi, \theta), \ -Q(\varphi, \theta)\} \tag{2.17}$$

$$Q(\varphi, \theta) = \frac{2}{K^2}\operatorname{sh}\frac{K}{(\gamma+1)W}\left[H\operatorname{sh}\frac{K}{(\gamma+1)W} - K\operatorname{ch}\frac{K}{(\gamma+1)W}\right] \tag{2.18}$$

for the case when $t^* < 0$, $W(\varphi, \theta) > 0$. Moreover $K = 1/\sqrt{R_1 R_2}$ and $H = \frac{1}{2}(1/R_1 + 1/R_2)$. Here K is the Gaussian curvature and H is the mean curvature of the surface S_0. It is assumed that, when the minimum is computed according to (2.17), the variables R_1, R_2 and W appearing in Q satisfy the relation

$$|\sqrt{R_1} - \sqrt{R_2}|\exp\frac{K}{(\gamma+1)W} - (\sqrt{R_1} + \sqrt{R_2})\exp\frac{-K}{(\gamma+1)W} \leqslant 0 \tag{2.19}$$

For $t^* > 0$ and $W < 0$ we obtain

$$t^* = \min_{\varphi,\theta} Q(\varphi, \theta) \tag{2.20}$$

If in (2.17) the minimum is reached when $|t^*| = R_i$, we have the effect of focussing the weak discontinuity on the plane of one of the principal normal cross sections of the surface R_t. The formulas (2.17) and (2.20) also yield the position of R_t, at which the potential flow begins to break down. In the one-dimensional spherical case when

$$R_1 = R_2 = 1/K = 1/H = R = \text{const}, \qquad W = \text{const}$$

we have from (2.17) and (2.20)

$$t^* = -R\left(1 - \exp\frac{-2}{(\gamma+1)WR}\right), \quad t^* < 0, \quad W > 0 \tag{2.21}$$

$$t^* = R\left(\exp\frac{2}{(\gamma+1)|W|R} - 1\right), \quad t^* > 0, \quad W < 0 \tag{2.22}$$

The case (2.21) corresponds to compression of gas contained at the initial instant within a sphere of radius R (piston moving into the sphere), and (2.22) corresponds to compression of gas by spherical piston expanding with a constant acceleration.

Passing in (2.21) and (2.22) to the limit with $R \to \infty$ (the case of a plane, one-dimensional flow), we obtain (1.1). If the passage to the limit is executed in (2.17), (2.19) and (2.20) by taking one of the R_i to infinity (this corresponds to the developable surfaces S_0), we obtain for t^* the formulas (1.21) and (1.22), i.e. the expression for t^* for the case of the developable surface S_0 formally coincides with the expression for t^* for the case of a plane parallel flow (differences in the magnitude of t^* may be caused by the differences in the radii of curvature in the plane and the three-dimensional case).

Note 2.1. The above construction using the solution of the transport equation on the bicharacteristics does not, unfortunately, allow us to obtain simple model equations near R_t as was done in Sect. 1. It is also difficult to obtain by this method the $o(r^2)$ order terms in the expression for $\Psi(r, \varphi, \theta, t)$ although this would be useful as the basic postulates of the Note 1.3 remain valid in the three-dimensional case.

Note 2.2. Inserting $t = t^*$ from (1.21), (1.22) and (2.17), (2.20) into the appropriate expressions for Ψ_t and using the formula (2.4) for c^2, we obtain the following expression for the density ρ in the perturbed motion:

$$\rho = \rho_0 \left(1 + r + \frac{3-\gamma}{4} r^2 + \cdots\right), \qquad \rho_0 = \text{const}$$

valid for both, the plane and the tree-dimensional case. Identical representation is obtained for ρ (up to the $o(r^2)$-order terms) from the Hugoniot conditions on the weak shock waves as well as in one-dimensional plane flows of the Riemann type wave [1]. This fact may play a very important role in the approximate investigation of propagation of three-dimensional weak shock waves appearing after the disruption of the potential flow in the piston problem just considered, under the assumption that the flow behind a weak shock wave remains potential (in the one-dimensional case in [1] the propagation of weak shock waves was studied using simple waves).

Note 2.3. The approximate formulas (1.19) and (2.14) obtained for the function Ψ can be used near R_t also in the case when the piston is withdrawn from the gas, when the potential flow will not be disrupted. Only the cases of $W < 0$ $t < 0$ and $W > 0$, $t > 0$ (up to the instant of focussing) should be considered.

BIBLIOGRAPHY

1. Courant, R. and Friedrichs, K. O., Supersonic Flow and Shock Waves. Interscience, N.Y., 1948.
2. Sidorov, A. F., On the gas flows near a weak shock. Dokl. Akad. Nauk SSSR, Vol. 167, №1, 1966.
3. Sidorov, A. F., Some three-dimensional gas flows adjacent to a region of rest. PMM Vol. 32, №3, 1968.
4. Sidorov, A. F., On nonstationary potential motions of a polytropic gas with a degenerate hodograph. PMM Vol. 23, №5, 1959.

Translated by L. K.